UNIT-IV.

DESIGN OF FIR DIGITAL FILTERS & REALIZATIONS

In many digital processing applications FIR filters are preferred over their IIR counterparts. The following are the main advantages of the FIR filter over IIR filter.

1. FIR filters are always stable
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive and non-recursive structures.
4. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital system.
5. Excellent design methods are available for various kinds of FIR filters

The disadvantages of FIR filter are

1. The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.
2. Memory requirement and execution time are very high.

FIR filters are employed in filtering problems where linear phase characteristics with in the pass band of the filter is required. If this is not required, either an FIR or an IIR filter may be employed.

An IIR filter has lesser no. of side lobes in the stop band than an FIR filter with the same no. of parameters. For this reason if some phase distortion is tolerable, an IIR filter is preferable. Also, the implementation of an IIR filter involves fewer parameters, less memory requirements and lower computational complexity.

→ Characteristics of FIR filters with Linear Phase

The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

where $h(n)$ is the impulse response of the filter

The Frequency response [Fourier transform of $h(n)$] is given by

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

which is periodic in frequency with period 2π, i.e

$$H(\omega) = H(\omega + 2k\pi) \quad , \quad k = 0, 1, 2 \ldots$$

Since $H(\omega)$ is complex it can be expressed as

$$H(\omega) = \pm |H(\omega)| e^{j\theta(\omega)}$$

where $|H(\omega)|$ is the magnitude response

$\theta(\omega)$ is the Phase response

We can define the phase delay 'τ_p' and group delay τ_g of a filter is

$$\tau_p = \frac{-\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g = \frac{-d\theta(\omega)}{d\omega}$$

For FIR filter with linear phase, We can define

$$\theta(\omega) = -\alpha\omega \qquad -\pi \leq \omega \leq \pi$$

where α is constant phase delay in samples

$$\tau_g = -\frac{d}{d\omega}\theta(\omega) \qquad \tau_p = \frac{-\theta(\omega)}{\omega} = \frac{-(-\alpha\omega)}{\omega} = \alpha$$

$$= \frac{-d}{d\omega}(-\alpha\omega) = \alpha$$

i.e $\tau_g = \tau_p = \alpha$ which means that 'α' is independent of frequency

We have

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(\omega)| e^{j\theta(\omega)}$$

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm|H(\omega)|[\cos\theta(\omega) + j\sin\theta(\omega)]$$

comparing the real and imaginary parts

$$\sum_{n=0}^{N-1} h(n)\cos\omega n = \pm|H(\omega)|\cos\theta(\omega)$$

$$-\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm|H(\omega)|\sin\theta(\omega)$$

$$\therefore \quad \frac{\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin\theta(\omega)}{\cos\theta(\omega)} = \frac{\sin\alpha\omega}{\cos\alpha\omega}$$

$$\sum_{n=0}^{N-1} h(n)[\sin\omega n\cos\alpha\omega - \cos\omega n\sin\alpha\omega] = 0$$

$$\sum_{n=0}^{N-1} h(n)\sin(\alpha-n)\omega = 0$$

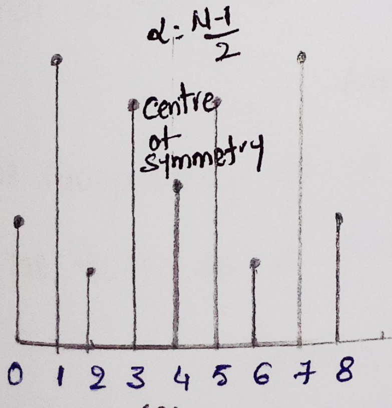
This will be zero when

$$h(n) = h(N-1-n) \quad \text{and} \quad \alpha = \frac{N-1}{2} \quad \text{for } 0 \le n \le N-1$$

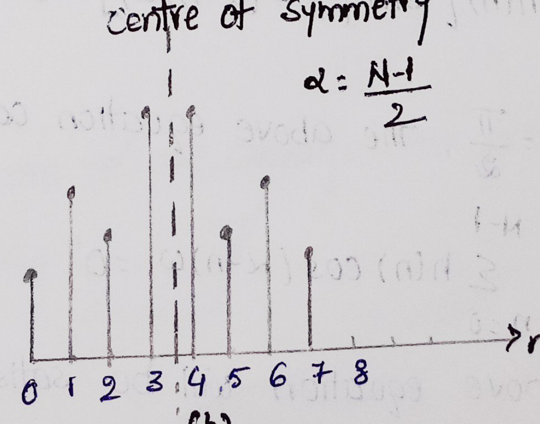
This shows that FIR filters will have constant phase and group delays when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$

The impulse response satisfying the symmetry condition $h(n) = h(N-1-n)$ for odd and even values of N is shown in fig 1.1.

When $N=9$, the centre of symmetry of the sequence occurs at the fourth sample and when $N=8$, the filter delay is $3\frac{1}{2}$ samples



(a)
'N' odd



(b)
'N' even

If only constant group delay is required and not the phase delay,

We can write

$$\theta(\omega) = \beta - \alpha\omega$$

Now, we have

$$H(\omega) = \pm |H(\omega)| e^{j(\beta-\alpha\omega)}$$

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(\omega)| e^{j(\beta-\alpha\omega)}$$

$$\sum_{n=0}^{N-1} h(n) [\cos\omega n - j\sin\omega n] = \pm |H(\omega)| [\cos(\beta-\alpha\omega) + j\sin(\beta-\alpha\omega)]$$

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm H(\omega) \cos(\beta-\alpha\omega)$$

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm H(\omega) \sin(\beta-\alpha\omega)$$

$$\therefore \frac{-\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)}$$

Cross multiplying and rearranging, we get

$$\sum_{n=0}^{N-1} h(n) [\sin\omega n \cos(\beta-\alpha\omega) + \cos\omega n \sin(\beta-\alpha\omega)] = 0$$

$$\sum_{n=0}^{N-1} h(n) [\sin(\omega n + \beta - \alpha\omega)] = 0$$

$$\sum_{n=0}^{N-1} h(n) [\sin(\beta - (\alpha - n)\omega)] = 0$$

If $\beta = \frac{\pi}{2}$, The above equation can be written as

$$\sum_{n=0}^{N-1} h(n) \cos(\alpha-n)\omega = 0$$

The above equation will be satisfied when

$$h(n) = -h(N-1-n) \quad \text{and} \quad \alpha = \frac{N-1}{2}$$

This shows that FIR filters have constant group delay τ_g and not constant phase delay when the impulse response is anti-symmetrical about $\alpha = \frac{N-1}{2}$

The impulse response satisfying the anti-symmetry condition is shown in Fig 1.2. When $N=9$, the centre of anti-symmetry occurs at fourth sample and when $N=8$, the centre of anti-symmetry occurs at $3\frac{1}{2}$ samples.

From fig 1.2, we find that $h[(\frac{N-1}{2})] = 0$ for antisymmetric odd sequence.

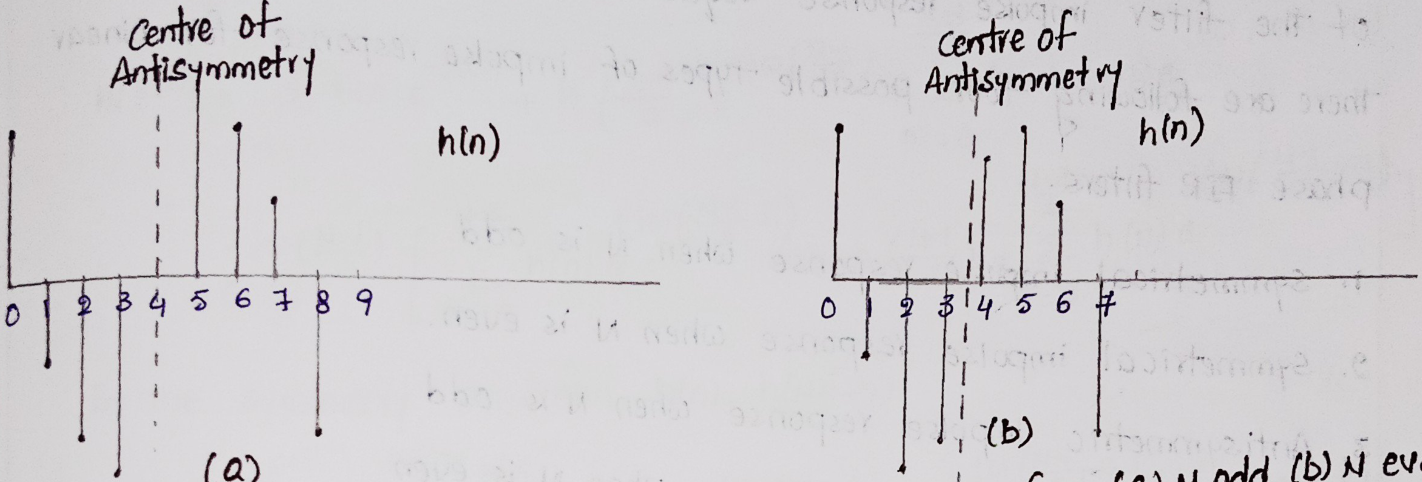


Fig 1.2: Impulse response of antisymmetric sequences for (a) N odd (b) N even

Phase delay: (τ_p)

The phase delay of the filter is the amount of time delay each frequency component of the signal suffers is going through the filter.

$$\tau_p = \frac{-\theta(\omega)}{\omega}$$

Group delay (τ_g)

The Group delay() is the average time delay the composite signal suffers at each frequency.

(or)

The Group delay of a filter is a measure of the average time delay of the filter as a function of frequency

$$\tau_g = \frac{-d}{d\omega}\theta(\omega)$$

→ Frequency Response of Linear Phase FIR Filters

The frequency response of the filter is the Fourier transform of its impulse response. If $h(n)$ is the impulse response of the system, then the frequency response of the system is denoted by $H(e^{j\omega})$ or $H(\omega)$. $H(\omega)$ is a complex function of frequency 'ω' and it can be expressed as magnitude function $|H(\omega)|$ and phase function $\angle H(\omega)$.

Depending on the value of N (odd or even) and the type of symmetry of the filter impulse response sequence (symmetric or Antisymmetric), there are following four possible types of impulse response for Linear phase FIR filters.

1. Symmetrical impulse response when N is odd.
2. Symmetrical impulse response when N is even.
3. Antisymmetric impulse response when N is odd
4. Antisymmetric impulse response when N is even

Frequency Response of Linear Phase FIR filter when impulse Response is symmetrical and 'N' is odd.

The frequency response of impulse response can be written as for N=6

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n)e^{-j\omega n} \quad — ①$$

This can be split like $H(e^{j\omega}) = \sum_{n=0}^{2} h(n)e^{-j\omega n} + h(3)e^{-j3\omega} + \sum_{n=4}^{6} h(n)e^{-j\omega n} \quad — ②$

In general, for N samples,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega(N-1)/2} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n} \quad — ③$$

Let $n = N-1-m$, we have

Put $$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(N-1-m)e^{-j\omega(N-1-m)} \quad — ④$$

put $n = \frac{N+1}{2}$ in $n = N-1-m$

$$\frac{N+1}{2} = N-1-m$$

$$m = N-1-\left(\frac{N+1}{2}\right) = \frac{2N-2-N-1}{2} = \frac{N-3}{2}$$

put $n = N-1$ in $n = N-1-m$

$$N-1 = N-1-m \Rightarrow m = 0.$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=\frac{N-3}{2}}^{0} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

change the variable 'm' to 'n'

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{+j\omega\left(\frac{N-1}{2}-n\right)} + h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]$$

By the symmetry condition $h(n) = h(N-1-n)$

$$\therefore H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{\frac{N-3}{2}} h(n)\left\{e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right\} + h\left(\frac{N-1}{2}\right)\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{\frac{N-3}{2}} h(n).\, 2\cos\omega\left(\frac{N-1}{2}-n\right) + h\left(\frac{N-1}{2}\right)\right]$$

Let $\frac{N-1}{2} - n = p$, then

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{p=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-p\right)\cos\omega p + h\left(\frac{N-1}{2}\right)\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

where $a(0) = h\left(\frac{N-1}{2}\right)$

$$a(n) = 2h\left(\frac{N-1}{2}-n\right)$$

We can write

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\,\bar{H}(e^{j\omega})$$

$$= e^{j\theta(\omega)}\,\bar{H}(e^{j\omega})$$

where $\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega$$

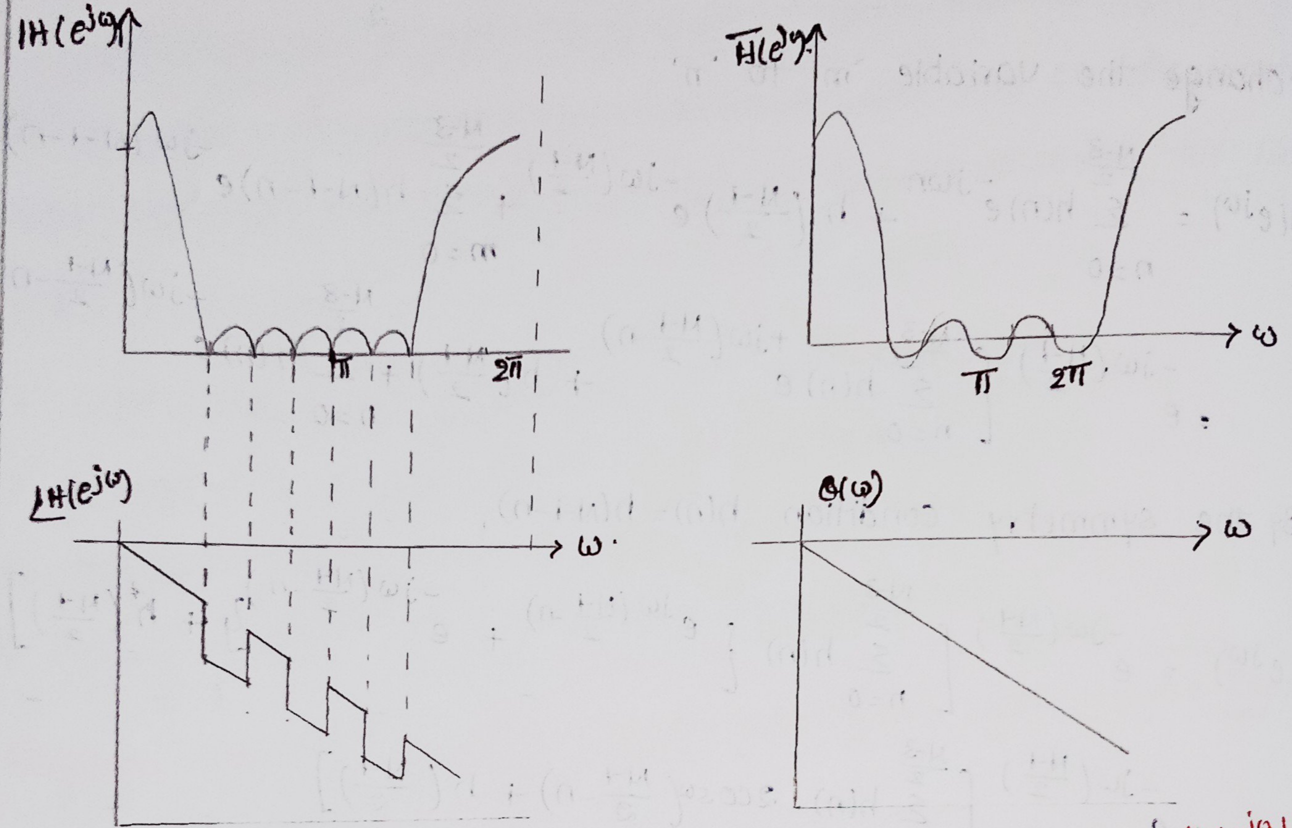


Fig: 1.8: Relation between magnitude response $|H(e^{j\omega})|$ and the phase response $\bar{H}(e^{j\omega})$ and between $\angle H(e^{j\omega})$ & $\theta(\omega)$

→ Determine the frequency response of FIR filter defined by $y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$. Calculate the phase delay and Group delay

Sol. Given $y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$

Taking Fourier transform on both sides

$$Y(e^{j\omega}) = 0.25X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) + 0.25e^{-j2\omega}X(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega})\left[0.25 + e^{-j\omega} + 0.25e^{-j2\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega}(0.25e^{j\omega} + 1 + 0.25e^{-j\omega})$$

$$= e^{-j\omega}(1 + 0.5\cos\omega)$$

$$H(e^{j\omega}) = e^{-j\omega}\bar{H}(e^{j\omega})$$

Comparing the above equation $H(e^{j\omega}) = e^{j\theta(\omega)}\bar{H}(e^{j\omega})$, we get

$$\theta(\omega) = -\omega$$

The Phase delay $\tau_p = \frac{-\theta(\omega)}{\omega} = \frac{-(-\omega)}{\omega} = 1$

The Group delay $\tau_g = \frac{-d}{d\omega}\theta(\omega) = \frac{-d}{d\omega}(-\omega) = 1$

→ The length of an FIR filter is 7. If this filter has a linear phase, show that the equation $\sum_{n=0}^{N-1} h(n)\sin(\alpha-n)\omega = 0$ is satisfied.

Sol: The length of the filter is 7. Therefore, for linear phase

$$\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

The condition for symmetry when N is odd, is $h(n) = h(N-1-n)$

Therefore, the filter co-efficients are $h(0) = h(6)$, $h(1) = h(5)$, $h(2) = h(4)$, $h(3)$

$$\therefore \sum_{n=0}^{N-1} h(n)\sin(\alpha-n)\omega = \sum_{n=0}^{6} h(n)\sin(3-n)\omega$$

$$= h(0)\sin 3\omega + h(1)\sin 2\omega + h(2)\sin\omega + h(3)\sin 0$$

$$+ h(4)\sin(-\omega) + h(5)\sin(-2\omega) + h(6)\sin(-3\omega)$$

$$= h(0)\sin 3\omega + h(1)\sin 2\omega + h(2)\sin\omega + 0 - h(2)\sin\omega$$

$$- h(1)\sin 2\omega - h(0)\sin 3\omega$$

$$= 0$$

Hence the equation $\sum_{n=0}^{N-1} h(n)\sin(\alpha-n)\omega = 0$ is satisfied.

→ The DESIGN TECHNIQUES FOR FIR FILTERS

The Well known methods of designing FIR filters are as follows:

1. Fourier series method
2. Window Method
3. Frequency sampling method
4. Optimum filter design method.

→ The Fourier Series Method of Designing FIR Filters

The desired frequency response of an FIR filter can be represented by the Fourier series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n} \quad -①$$

where the Fourier co-efficients $h_d(n)$ are the desired impulse response sequence of the filter

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega \quad -②$$

The z-transform of the sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n) z^{-n} \quad -③$$

The eq ③ represents a non-causal digital filter of infinite duration. To get an FIR filter transfer function, the series can be truncated by assigning

$$h(n) = \begin{cases} h_d(n) & \text{for } |n| \le \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases} \quad -④$$

Then eq ③ becomes

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h_d(n) z^{-n} \quad -⑤$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \dots + h(1)z^{-1} + h(0) + h(-1)z + h(-2)z^{2} + \dots + h\left[-\left(\frac{N-1}{2}\right)\right] z^{\frac{N-1}{2}}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n) z^{-n} + h(-n) z^{n} \right] \quad - (6)$$

For a symmetrical impulse response having symmetry at $n=0$

$$h(-n) = h(n) \quad - (7)$$

substitute eq (7) in (6) then, we get

$$H(Z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n) z^{-n} + h(n) z^{-n} \right]$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right] \quad - (8)$$

The above transfer function is not physically realizable. Realizability can be brought by multiplying eq (8) by $z^{-(N-1/2)}$ where $\frac{N-1}{2}$ is delay in samples.

$$H'(z) = z^{-(N-1/2)} H(z)$$

$$H(z) = z^{-(N-1/2)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) (z^{n} + z^{-n}) \right] \quad - (9)$$

→ Design an ideal low pass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } -\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$$

$$= 0 \quad \text{for } \frac{\pi}{2} \le |\omega| \le \pi$$

Find the values of $h(n)$ for $N=11$. Find $H(Z)$. Plot the magnitude response.

Sol. The frequency response of LPF with $\omega_c = \frac{\pi}{2}$ is shown in fig

Given $H_d(e^{j\omega}) = 1$ for $-\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$

$= 0$ for $\frac{\pi}{2} \le |\omega| \le \pi$

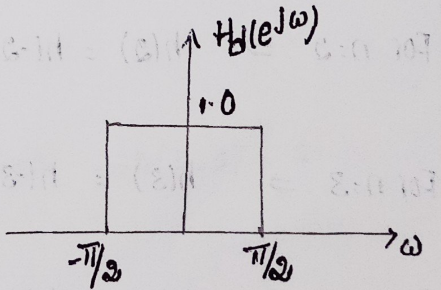


Fig 1.4 Ideal frequency response of given example

From the frequency response we can find that $\alpha = 0$

Therefore, we get a non-causal filter co-efficients symmetrical about $n=0$

$h_d(n) = h_d(-n)$

The filter co-efficients can be obtained by using the formula

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/2}^{\pi/2}.$$

$$= \frac{1}{n\pi}\left[\frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j}\right]$$

$$h_d(n) = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) \qquad -\infty \le n \le \infty$$

Truncating $h_d(n)$ to 11 samples we have

$$h(n) = \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi} \quad \text{for } |n| \le 5$$

$$= 0 \qquad \text{otherwise}$$

For $n=0$ the eq becomes indeterminate. so L-hospital rule

$$h(0) = \underset{n\to 0}{Lt}\ \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$= \frac{1}{2}\cdot \underset{n\to 0}{Lt}\ \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} \qquad \left[\because \underset{n\to 0}{Lt}\ \frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{1}{2}$$

For $n=1 \Rightarrow h(1) = \frac{\sin\frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.3183 = h(-1)$

For $n=2 \Rightarrow h(2) = h(-2) = \frac{\sin\frac{2\pi}{2}}{2\pi} = \frac{\sin\pi}{2\pi} = 0$

For $n=3 \Rightarrow h(3) = h(-3) = \frac{\sin\frac{3\pi}{2}}{3\pi} = \frac{-1}{3\pi} = -0.106$

For $n=4 \Rightarrow h(4) = h(-4) = \frac{\sin 2\pi}{4\pi} = 0$

For $n=5 \Rightarrow h(5) = h(-5) = \frac{\sin\frac{5\pi}{2}}{5\pi} = \frac{1}{5\pi} = 0.06366$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} [h(n)(z^n+z^{-n})]$$

$$= 0.5 + \sum_{n=1}^{5} h(n)[z^n+z^{-n}]$$

$$= 0.5 + h(1)(z+z^{-1}) + h(2)(z^2+z^{-2}) + h(3)(z^3+z^{-3}) + h(4)(z^4+z^{-4})$$
$$+ h(5)(z^5+z^{-5})$$

$$H(z) = 0.5 + 0.3183(z+z^{-1}) - 0.106(z^3+z^{-3}) + 0.06366(z^5+z^{-5})$$

The transfer function of the realizable filter is

$$H'(z) = z^{-(N-1)/2} H(z)$$

$$= z^{-5}[0.5 + 0.3183(z+z^{-1}) - 0.106(z^3+z^{-3}) + 0.06366(z^5+z^{-5})]$$

$$H'(z) = 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8}$$
$$+ 0.06366z^{-10}$$

From the above equation the filter co-efficients of causal filter are given by

$h(0) = h(10) = 0.06366$; $h(1) = h(9) = 0$; $h(2) = h(8) = -0.106$

$h(3) = h(7) = 0$; $h(4) = h(6) = 0.3183$; $h(5) = 0.5$

The frequency response is given by

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n$$

$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.5$

$a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$a(1) = 2h(5-1) = 2h(4) = 2 \times 0.3183 = 0.6366$

$a(2) = 2h(5-2) = 2h(3) = 2 \times 0 = 0$

$a(3) = 2h(5-3) = 2h(2) = 2 \times (-0.106) = -0.212$

$a(4) = 2h(5-4) = 2h(1) = 0$

$a(5) = 2h(5-5) = 2h(0)$
$= 0.127$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n$$

$$= a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega$$

$$\bar{H}(e^{j\omega}) = 0.5 + 0.6366\cos\omega - 0.212\cos 3\omega + 0.127\cos 5\omega$$

The magnitude in dB is calculated by varying ω from 0 to π and tabulated below. The magnitude $|H(e^{j\omega})|_{dB} = 20\log|\bar{H}(e^{j\omega})|$

ω (in degree)	0	10	20	30	40	50	60	70	80	90	100
$\vert H(e^{j\omega})\vert_{dB}$	0.4	0.21	-0.26	-0.517	-0.21	0.42	0.77	0.21	-1.79	-6	-14.56

110	120	130	140	150	160	170	180
~~-6~~	~~-14.56~~	~~-31.89~~	~~-20.6~~	~~-26~~	~~-32~~	~~-24.7~~	-
-31.89	-20.6	-26	-32	-24.7	-30.55	-32	-26

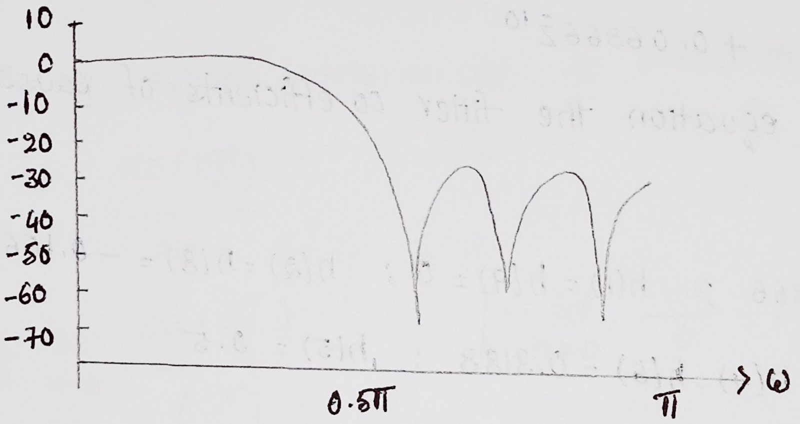


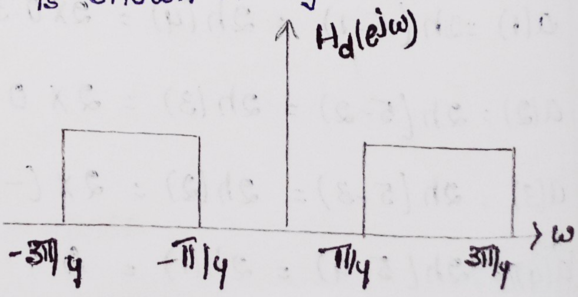
Fig 1.5: Frequency response of LPF of given example

→ Design an ideal band pass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$$
$$= 0 \quad \text{otherwise}$$

Find the values of $h(n)$ for $N = 11$ and plot the frequency response.

Sol. The ideal frequency response of the filter is shown in figure



The ideal frequency response

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-3\pi/4}^{-\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-3\pi/4}^{-\pi/4} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{1}{2\pi nj}\left[e^{-jn\pi/4} - e^{-3\pi/4\, jn}\right] + \frac{1}{2\pi nj}\left[e^{j\frac{3\pi}{4}n} - e^{+j n\pi/4}\right]$$

$$= \frac{1}{2\pi nj}\left[e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n} \mp e^{jn\pi/4} - e^{-jn\pi/4}\right]$$

$$= \frac{1}{n\pi}\left[\sin\frac{3\pi}{4}n \mp \sin\frac{n\pi}{4}\right] \quad -\infty \le n \le \infty$$

Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = h_d(n) \quad \text{for } |n| \le 5$$

$$= 0 \quad \text{otherwise}$$

The filter co-efficients are symmetrical about $n=0$ satisfying the conditions $h(n) = h(-n)$

For $n=0$

$$h(0) = \frac{1}{2\pi}\left[\int_{-3\pi/4}^{-\pi/4} d\omega + \int_{\pi/4}^{3\pi/4} d\omega\right]$$

$$= \frac{1}{2\pi}\left[(\omega)_{-3\pi/4}^{-\pi/4} + (\omega)_{\pi/4}^{3\pi/4}\right] = \frac{1}{2\pi}\left[\frac{-\pi}{4} + \frac{3\pi}{4} + \frac{3\pi}{4} - \frac{\pi}{4}\right]$$

$$= \frac{4\pi}{2\pi \times 4} = \frac{1}{2} = 0.5$$

$$h(1) = h(-1) = \frac{1}{\pi}\left[\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}\right] = \frac{0.707 - 0.707}{\pi} = 0$$

$$h(2) = h(-2) = \frac{\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}}{2\pi} = \frac{-2}{2\pi} = -0.3183$$

$$h(3) = h(-3) = \frac{\sin\frac{9\pi}{4} - \sin\frac{3\pi}{4}}{3\pi} = 0$$

$$h(4) = h(-4) = \frac{\sin 3\pi - \sin\pi}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin\frac{15\pi}{4} - \sin\frac{5\pi}{4}}{5\pi} = 0$$

The transfer function of the filter is :

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} [\, h(n)\,(z^n + z^{-n})]$$

$$= 0.5 - 0.3183\,(z^2 + z^{-2})$$

The transfer function of the realizable filter is

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$$

$$= z^{-5}[\,0.5 - 0.3183\,(z^2 + z^{-2})]$$

$$= 0.5z^{-5} - 0.3183\,z^{-3} - 0.3183\,z^{-7}$$

The filter co-efficients of the causal filters are

$h(0) = h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$

$h(3) = h(7) = -0.3183$

$h(5) = 0.5$

$$\bar{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n$$

$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.5$

$a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$a(1) = 2h(5-1) = 2h(4) = 0$

$a(2) = 2h(5-2) = 2h(3) = -0.6366$

$a(3) = 2h(5-3) = 2h(2) = 0$

$a(4) = 2h(5-4) = 2h(1) = 0$

$a(5) = 2h(5-5) = 2h(0) = 0$

$\bar{H}(e^{j\omega}) = 0.5 - 0.6366\cos 2\omega$

ω in degrees	0	20	30	45	60	75	90	105	120	135	150	160
$\bar{H}(e^{j\omega})$	-0.1366	0.012	0.1817	0.5	0.818	1.05	1.1366	1.05	0.818	0.5	0.1817	0.012
$\lvert H(e^{j\omega})\rvert$ dB	-17.3	-38.17	14.8	-6.02	-1.74	0.4346	1.11	0.4346	-1.74	-6.02	-14.8	-38.17

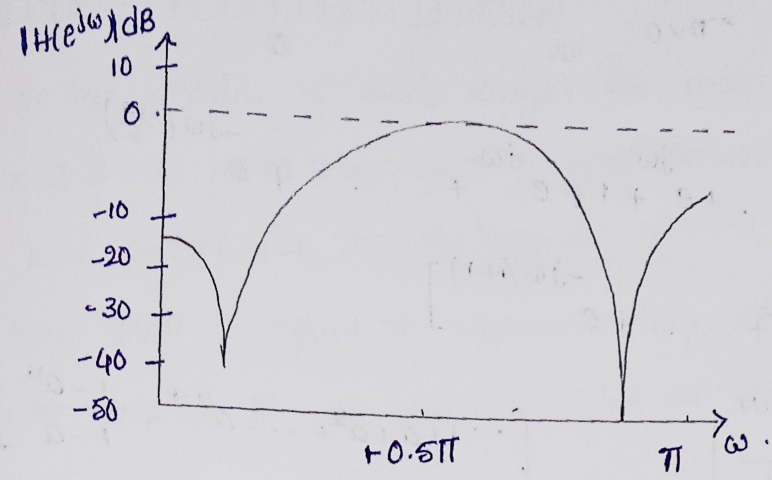


Fig 1.6: Frequency response of Band Pass filter of given example.

→ DESIGN OF FIR FILTERS USING WINDOWS

The Procedure for designing FIR filter using window is

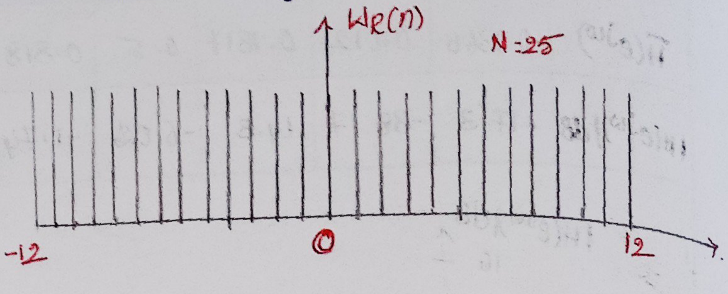
1. choose the desired frequency response of the filter $H_d(\omega)$
2. Take inverse Fourier transform of $H_d(\omega)$ to obtain the desired impulse response $h_d(n)$
3. choose a window sequence $w(n)$ and multiply $h_d(n)$ by $w(n)$ to convert the infinite duration impulse response to a finite duration impulse response $h(n)$
4. The transfer function $H(z)$ of the filter is obtained by taking z-transform of $h(n)$.

→ Rectangular Window

The Weighting function (Window function) for an N-point rectangular window is given by

$$W_R(n) = \begin{cases} 1, & \text{for } -\frac{(N-1)}{2} \le n \le \frac{N-1}{2} \\ 0, & \text{elsewhere} \end{cases} \quad \text{(or)} \quad W_R(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

The spectrum of rectangular window $W_R(\omega)$ is given by the Fourier transform of $W_R(n)$



$$W_R(\omega) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\omega n} = \sum_{n=0}^{N-1} e$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} + \dots + e^{j\omega} + 1 + e^{-j\omega} + \dots + e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[1 + e^{-j\omega} + \dots + e^{-j\omega(N-1)}\right]$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right] \qquad \left[\because 1 + a + a^2 + \dots a^{N-1} = \frac{1-a^N}{1-a}\right]$$

$$= \frac{e^{j\omega N/2}\left(1-e^{-j\omega N}\right)}{e^{j\omega/2}\left(1-e^{-j\omega}\right)} = \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} = \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

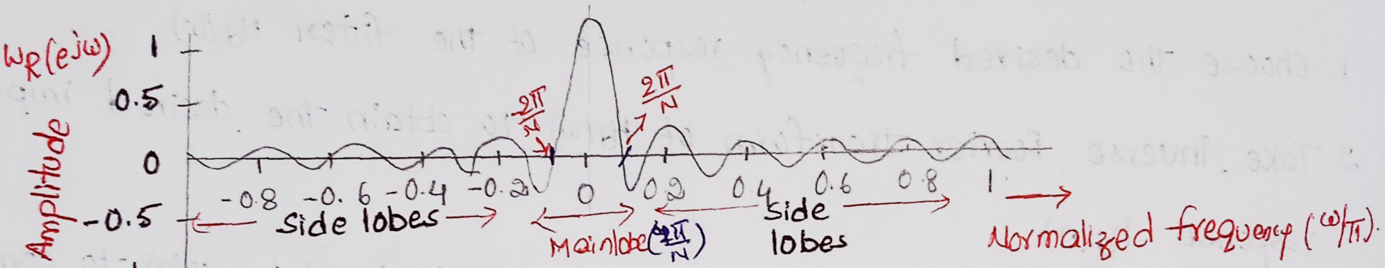


Fig7(a): Frequency Response of rectangular window N=25

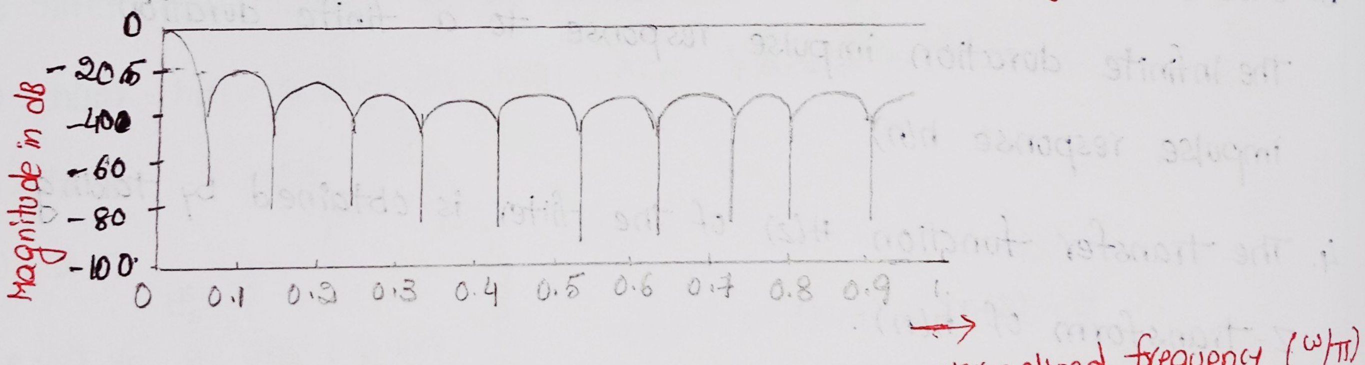


Fig8(b): Log magnitude response of rectangular window

The frequency response is real and its zero when $\frac{N\omega}{2} = k\pi$ or $\omega = \frac{2\pi k}{N}$, where k is an integer. The response for ω between $\frac{2\pi}{N}$ and $-\frac{2\pi}{N}$ is called the main lobe and the other lobes are known as side lobes.

The main lobe of the response is the portion that lies between the first two zero crossings. The side lobes are defined as the position that ~~lie between the first two zero crossings. The~~ side portion of the response for $\omega < -\frac{2\pi}{N}$ or $\omega > \frac{2\pi}{N}$.

As the window is made longer the main lobe becomes narrower and higher and the side lobe become more concentrated around $\omega = 0$. The main lobe width for the rectangular window is equal to $\frac{4\pi}{N}$. The higher side lobe level is equal to approximately 22% of the main lobe amplitude or -13dB relative to the maximum value at $\omega = 0$ as shown in fig 1.7(b) & (a).

If we design a LPF using rectangular window, we find that the frequency response differs from the desired frequency response in many ways. It does not follow quick transitions in the desired response.

The desired response of a LPF changes abruptly from passband to stop band, but the actual frequency response changes slowly This region of gradual change is called filter's transition region, which is due to the convolution of the desired response with the window response's main lobe.

The width of the transition region depends on the width of the main lobe. As the filter N increases, the main lobe becomes narrower decreasing the width of the transition region.

The convolution of the desired response and the window response's side lobe give rise to the ripples in both the pass band and stop band. The amplitude of the ripples is dictated by the amplitude of side lobes. This effect, where maximum ripple occurs just before and just after the transition band, is known as Gibb's phenomeon

The Gibbs phenomeon can be reduced by using a less obrupt truncation of filter co-efficients. This can be achieved by using a window function that tapers smoothly towards zero at both ends.

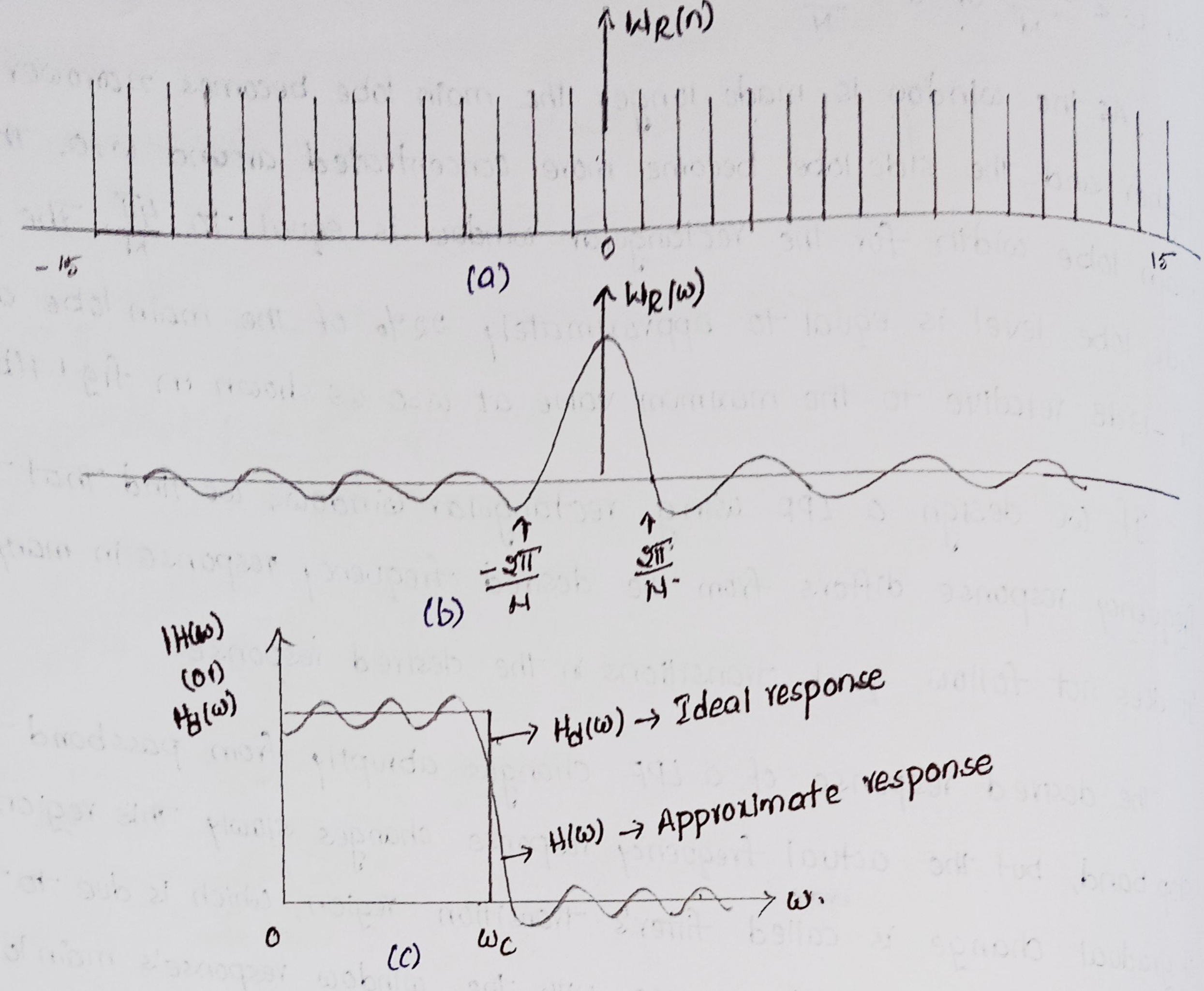


Fig 1.9(a) : Rectangular Window sequence

(b) : Magnitude response of rectangular window

(c) : Magnitude response of LPF approximated using rectangular window

→ Triangular or Barlett Window

The triangular window has been chosen such that it has tapered sequences from the middle on either side. The window function $W_T(n)$ is defined as

$$W_T(n) = \begin{cases} 1 - \frac{2|n|}{N-1}, & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

$$W_T(n) = \begin{cases} \frac{1 - 2|n - (N-1)/2|}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

In magnitude response of triangular window, the side lobe level is smaller than that of the rectangular window being reduced from -13 dB to -25 dB. However, the main lobe width is now $\frac{8\pi}{N}$ or twice that of the rectangular window.

The triangular window produces a smooth magnitude response in both pass band and stop band, but it has the following disadvantages when compared to magnitude response obtained by using rectangular window

1. The transition region is more
2. The attenuation in stop band is less.

Because of these characteristics, the triangular window is not usually a good choice.

→ Raised Cosine Window

The raised cosine window multiplies the central Fourier co-efficients by approximately unity and smoothly truncates the Fourier co-efficients towards the ends of the filter. The smoother ends and broader middle section produces less distortion of $h_d(n)$ around $n=0$. It is also called generalized Hamming window

$$W_H(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right), & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases}$$

→ Hanning Window

The Hanning window function is given by

$$W_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\left(\frac{2\pi n}{N-1}\right), & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

$$W_H(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & \text{for } 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

The width of the main lobe is $\frac{8\pi}{N}$, i.e twice that of rectangular window which results in doubling of the transition region of the filter. The peak of the first lobe is -32 dB relative to the maximum value. This results in smaller ripples in both pass band and stop band of the low-pass filter designed using Hanning window.

The minimum stop band attenuation of the filter is 44dB. At higher frequencies the stop band attenuation is even greater.

When compared to triangular window, the main lobe width is same, but the magnitude of the side lobe is reduced, hence the Hanning window is preferable to triangular window.

Hamming Window

The Hamming window function is given by

$$W_H(n) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 \end{cases}$$

In the magnitude response for $N=31$, the magnitude of the first side lobe is down by about 41 dB from the main lobe peak, an improvement of 10 dB relative to the Hanning window.

The width of the main lobe is $\frac{8\pi}{N}$. In the magnitude response of LPF designed using Hamming window, the first side lobe peak is -51dB, which is -7dB lesses with respect to the Hanning window filter.

Hamming window generates lesser oscillations in the side lobes than the Hanning window for the same main lobe width, the Hamming window is generally preferred.

$$W_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Blackman Window

The Blackman Window function is another type of cosine window and given by the equation

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{, otherwise} \end{cases}$$

In the magnitude response, the width of the main lobe is $12\pi/N$, which is highest among windows. The peak of the first side lobe is at -58dB and the side lobe magnitude decreases with frequency. This desirable feature is achieved at the expense of increased main lobe width. However the main lobe width can be reduced by increasing the value of N. The side lobe attenuation of a LPF using Blackman window is -78 dB.

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2n\pi}{N-1}\right) + 0.08\cos\frac{4\pi n}{N-1}, & 0 \le n \le N-1 \\ 0 & \text{, otherwise} \end{cases}$$

	Type of Window	Approximate transition width of Main lobe	Minimum stop band attenuation (dB)	Peak of first side lobe (dB)
1.	Rectangular	$4\pi/N$	-21	-13
2.	Barlett	$8\pi/N$	-25	-25
3.	Hanning	$8\pi/N$	-44	-31
4.	Hamming	$8\pi/N$	-51	-41
5.	Blackmann	$12\pi/N$	-78	-58

Table: Frequency domain characteristics of some window functions

→ Design an ideal Low-pass filter with $N=11$ with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & \text{for } \frac{-\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0, & \text{for } \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

sol. For the given desired frequency response,

$$H_d(\omega) = \begin{cases} 1, & \text{for } \frac{-\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0, & \text{for } \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

The filter co-efficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/2}^{\pi/2}$$

$$\frac{1}{2n\pi j}\left[e^{jn\pi/2} - e^{-jn\pi/2}\right] = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) \text{ for } n \ne 0.$$

$$h_d(0) = \underset{n\to 0}{Lt}\ \frac{1}{2}\cdot\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} = \frac{1}{2} \quad \text{[using L-Hospital rule]}$$

$$h_d(1) = \frac{1}{\pi}\sin\frac{\pi}{2} = \frac{\pi}{2} = h_d(-1)$$

$$h_d(2) = \frac{1}{2\pi}\sin\pi = 0 = h_d(-2)$$

$$h_d(3) = \frac{1}{3\pi}\sin\left(\frac{3\pi}{2}\right) = \frac{-1}{3\pi} = h_d(-3)$$

$$h_d(4) = \frac{1}{4\pi}\sin(2\pi) = 0 = h_d(-4)$$

$$h_d(5) = \frac{1}{5\pi}\sin\left(\frac{5\pi}{2}\right) = \frac{1}{5\pi} = h_d(-5).$$

Assuming the window function

$$w(n) = \begin{cases} 1 & \text{for } -5 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$$

We have $h(n) = h_d(n) \cdot w(n) = h_d(n)$

Therefore, the designed filter co-efficients are given as

$h(0) = \frac{1}{2}$, $h(1) = h(-1) = \frac{1}{\pi}$, $h(2) = h(-2) = 0$, $h(3) = h(-3) = \frac{-1}{3\pi}$

$h(4) = h(-4) = 0$, $h(5) = h(-5) = \frac{1}{5\pi}$

The above co-efficients correspond to a non-causal filter which is not realizable. The realizable digital filter transfer function $H(z)$ is given by

$$H(z) = z^{-(N-1/2)} \left[h(0) + \sum_{n=1}^{\left(\frac{N-1}{2}\right)} h(n)\left[z^n + z^{-n}\right] \right\}$$

$$= z^{-5} \left\{ h(0) + \sum_{n=1}^{5} h(n)\left[z^n + z^{-n}\right] \right\}$$

$$= z^{-5} \left[\frac{1}{2} + \frac{1}{\pi}(z + z^{-1}) - \frac{1}{3\pi}(z^3 + z^{-3}) + \frac{1}{5\pi}(z^5 + z^{-5}) \right]$$

$$= h(5) + h(3)z^{-2} + h(1)z^{-4} + h(0)z^{-5} + h(1)z^{-6} + h(3)z^{-8} + h(5)z^{-10}$$

$$= \frac{1}{5\pi} - \frac{1}{3\pi}z^{-2} + \frac{1}{\pi}z^{-4} + \frac{1}{2}z^{-5} + \frac{1}{\pi}z^{-6} - \frac{1}{3\pi}z^{-8} + \frac{1}{5\pi}z^{-10}$$

Therefor the co-efficients of the realizable digital filter are

$h(0) = \frac{1}{5\pi} = h(10)$, $h(1) = 0 = h(9)$, $h(2) = \frac{-1}{3\pi} = h(8)$, $h(3) = 0 = h(7)$,

$h(4) = \frac{1}{\pi} = h(6)$, $h(5) = \frac{1}{2}$.

→ Design a filter with $H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & , \frac{-\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & , \frac{\pi}{4} \le \omega \le \pi \end{cases}$ using a Hamming window with $N=7$.

Sol. For the given filter with $H_d(\omega) = \begin{cases} e^{-j3\omega} & , \frac{-\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & , \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$

The filter co-efficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} \cdot d\omega = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j3\omega} \cdot e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-3)} \, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-3)}}{j(n-3)}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi(n-3)}\left[\frac{e^{j(n-3)\pi/4} - e^{-j(n-3)\pi/4}}{2j}\right]$$

$$= \frac{1}{\pi(n-3)} \cdot \sin(n-3)\pi/4 = \frac{\sin(n-3)\frac{\pi}{4}}{\pi(n-3)}, \quad n \neq 3$$

For $n=3$, the filter co-efficient can be obtained by applying L-Hospital rule to the above expression. Thus

$$h_d(3) = \underset{n\to 3}{Lt} \frac{\sin(n-3)\frac{\pi}{4}}{(n-3)\frac{\pi}{4}} \cdot \frac{1}{4} = \frac{1}{4} \qquad \left[\because \underset{\theta\to 0}{Lt} \frac{\sin\theta}{\theta} = 1\right]$$

The other filter co-efficients are given by

$$h_d(0) = \frac{\sin\pi(0-3)/4}{\pi(0-3)} = \frac{0.707}{3\pi}$$

$$h_d(1) = \frac{\sin\pi(1-3)/4}{\pi(1-3)} = \frac{-\sin\pi/2}{-2\pi} = \frac{1}{2\pi}$$

$$h_d(2) = \frac{\sin\pi(2-3)/4}{\pi(2-3)} = \frac{-\sin\pi/4}{-\pi} = \frac{0.707}{\pi}$$

$$h_d(4) = \frac{\sin\pi(4-3)/4}{\pi(4-3)} = \frac{\sin\pi/4}{\pi} = \frac{0.707}{\pi}$$

$$h_d(5) = \frac{\sin\pi(5-3)/4}{\pi(5-3)} = \frac{\sin\pi/2}{2\pi} = \frac{1}{2\pi}$$

$$h_d(6) = \frac{\sin\pi(6-3)/4}{\pi(6-3)} = \frac{\sin 3\pi/4}{3\pi} = \frac{0.707}{3\pi}$$

So, the filter co-efficients are

$h_d(0) = \frac{0.707}{3\pi}$, $h_d(1) = \frac{1}{2\pi}$, $h_d(4) = \frac{0.707}{\pi}$

$h_d(2) = \frac{0.707}{\pi}$, $h_d(3) = \frac{1}{4}$, $h_d(5) = \frac{1}{2\pi}$, $h_d(6) = \frac{0.707}{3\pi}$

The Hamming window function of a causal filter is

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{Otherwise} \end{cases}$$

Therefore with $N=7$,

$w(0) = 0.54 - 0.46\cos 0 = 0.08$

$w(1) = 0.54 - 0.46\cos\left(\frac{2\pi \times 1}{7-1}\right) = 0.31$

$w(2) = 0.54 - 0.46\cos\left(\frac{2\pi \times 2}{7-1}\right) = 0.77$

$w(3) = 0.54 - 0.46\cos\left(\frac{2\pi \times 3}{7-1}\right) = 1$

$w(4) = 0.54 - 0.46\cos\left(\frac{2\pi \times 4}{7-1}\right) = 0.77$

$w(5) = 0.54 - 0.46\cos\left(\frac{2\pi \times 5}{7-1}\right) = 0.31$

$w(6) = 0.54 - 0.46\cos\left(\frac{2\pi \times 6}{7-1}\right) = 0.08$

The filter co-efficients of the resultant filter are

$$h(n) = h_d(n)\,w(n), \quad n = 0, 1, 2, 3, 4, 5, 6.$$

Therefore

$h(0) = h_d(0)\,w(0) = \frac{0.707}{3\pi} \times 0.08 = 0.006$

$h(1) = h_d(1)\,w(1) = \frac{1}{2\pi} \times 0.31 = 0.049$

$h(2) = h_d(2)\,w(2) = \frac{0.707}{\pi} \times 0.77 = 0.173$

$h(3) = h_d(3)\,w(3) = \frac{1}{4} \times 1 = \frac{1}{4} = 0.25$

$h(4) = h_d(4)\,w(4) = \frac{0.707}{\pi} \times 0.77 = 0.173$

$h(5) = h_d(5)\,w(5) = \frac{1}{2\pi} \times 0.31 = 0.049$

$h(6) = h_d(6)\,w(6) = \frac{0.707}{3\pi} \times 0.08 = 0.006$

The frequency response of a causal filter is given by

$$H(\omega) = \sum_{n=0}^{6} h(n) e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega} + h(5)e^{-j5\omega} + h(6)e^{-j6\omega}$$

$$= e^{-j3\omega}\left[h(3) + [h(0)e^{j3\omega} + h(6)e^{-j3\omega}] + [h(1)e^{j2\omega} + h(5)e^{-j2\omega}]\right.$$

$$\left. + [h(2)e^{j\omega} + h(4)e^{-j\omega}]\right]$$

$$= e^{-j3\omega}\left[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega\right]$$

$$= e^{-j3\omega}\left[0.25 + 0.012\cos 3\omega + 0.098\cos 2\omega + 0.346\cos\omega\right]$$

The transfer function of the digital FIR low-pass filter is

$$H(Z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{6} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= z^{-3}\left[h(3) + h(2)(z^{-1}+z) + h(1)(z^{-2}+z^{2}) + h(0)(z^{-3}+z^{3})\right]$$

$$= z^{-3}\left[0.25 + 0.173(z+z^{-1}) + 0.049(z^{2}+z^{-2}) + 0.006(z^{3}+z^{-3})\right]$$

$$= 0.006 + 0.049z^{-1} + 0.173z^{-2} + 0.25z^{-3} + 0.173z^{-4} + 0.049z^{-5} + 0.006z^{-6}$$

→ Design a high-pass filter using Hamming window, with a cut-off frequency of 1.2 rad/sec and N = 9.

Sol. The desired frequency response $H_d(\omega)$ for a high-pass filter is

$$H_d(\omega) = \begin{cases} 0, & -\omega_c \le \omega \le \omega_c \\ e^{-j\omega\alpha}, & \omega_c \le |\omega| \le \pi \end{cases}$$

The desired impulse response $h_d(n)$ is obtained by taking the inverse Fourier transform of $H_d(\omega)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\alpha} \cdot e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)}\,d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(n-\alpha)\omega_c} - e^{-j(n-\alpha)\pi} + e^{j(n-\alpha)\pi} - e^{j(n-\alpha)\omega_c}}{j(n-\alpha)}\right]$$

$$= \frac{1}{(n-\alpha)\pi}\left[\frac{e^{j(n-\alpha)\pi} - e^{-j(n-\alpha)\pi}}{2j} - \frac{e^{j(n-\alpha)\omega_c} - e^{-j(n-\alpha)\omega_c}}{2j}\right]$$

$$= \frac{1}{(n-\alpha)\pi}\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c\right]$$

When $n=\alpha$, the terms $\frac{\sin(n-\alpha)\pi}{\pi(n-\alpha)}$ and $\frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$ become $0/0$ which is indeterminate form.

Using 'L'-Hospital rule, we have for $n=\alpha$

$$h_d(n) = \underset{n\to\alpha}{Lt}\ \frac{1}{\pi}\left[\frac{\sin(n-\alpha)\pi}{(n-\alpha)} - \frac{\sin(n-\alpha)\omega_c}{(n-\alpha)}\right]$$

$$= \frac{1}{\pi}\left[\pi - \omega_c\right] = 1 - \frac{\omega_c}{\pi}$$

For $n \neq \alpha$

$$h_d(n) = \frac{\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c}{(n-\alpha)\pi}$$

Here $\alpha = \frac{N-1}{2} = \frac{9-1}{2} = \frac{8}{2} = 4$ is an integer and since 'n' is also an integer, $(n-\alpha)$ is also an integer and so $\sin(n-\alpha)\pi = 0$

$$h_d(n) = \frac{-\sin(n-\alpha)\omega_c}{(n-\alpha)\pi}$$

$$= \frac{-\sin(n-4)\omega_c}{(n-4)\pi}$$

$$h_d(0) = \frac{-\sin(0-4).1.2}{\pi(0-4)} = \frac{-\sin 4 \times 1.2}{-4\pi} = \frac{\cancel{0.0697 \times 1.2}}{4\pi} = \frac{\sin 4.8}{4\pi} = 0.0792$$

$$h_d(1) = -\frac{\sin(1-4)1.2}{\pi(1-4)} = \frac{-0.442}{-3\pi} = 0.0469$$

$$h_d(2) = -\frac{\sin(2-4)1.2}{\pi(2-4)} = -0.1075$$

$$h_d(3) = -\frac{\sin(3-4)1.2}{\pi(3-4)} = -0.2966$$

$$h_d(4) = 1 - \frac{\omega_c}{\pi_c} = 1 - \frac{1.2}{\pi} = 0.618$$

$$h_d(5) = -\frac{\sin(5-4)1.2}{\pi(5-4)} = -0.2966$$

$$h_d(6) = \frac{-\sin(6-4)1.2}{\pi(6-4)} = -0.1075$$

$$h_d(7) = \frac{-\sin(7-4)1.2}{\pi(7-4)} = -0.0469$$

$$h_d(8) = \frac{-\sin(8-4)1.2}{\pi(8-4)} = -0.0792.$$

The window sequence for Hamming window is given by

$$\omega_H(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$$

$$W_H(0) = 0.54 - 0.46\cos\left(\frac{2\pi\times 0}{9-1}\right) = 0.08$$

$$W_H(1) = 0.54 - 0.46\cos\left(\frac{2\pi\times 1}{9-1}\right) = 0.2147$$

$$\omega_H(2) = 0.54 - 0.46\cos\left(\frac{2\pi\times 2}{9-1}\right) = 0.54$$

$$\omega_H(3) = 0.54 - 0.46\cos\left(\frac{2\pi\times 3}{9-1}\right) = 0.8652$$

$$\omega_H(4) = 0.54 - 0.46\cos\left(\frac{2\pi\times 4}{9-1}\right) = 1$$

$$\omega_H(5) = 0.54 - 0.46\cos\left(\frac{2\pi\times 5}{9-1}\right) = 0.8652$$

$$\omega_H(6) = 0.54 - 0.46\cos\left(\frac{2\pi\times 6}{9-1}\right) = 0.54$$

$$W_H(7) = 0.54 - 0.46\cos\left(\frac{2\pi\times 7}{9-1}\right) = 0.2147$$

$$\omega_H(8) = 0.54 - 0.46\cos\left(\frac{2\pi\times 8}{9-1}\right) = 0.08$$

The filter co-efficients are $h(n) = h_d(n)\,\omega_H(n)$.

$$h(0) = h_d(0)\omega_H(0) = 0.0792 \times 0.08 = 0.0063$$

$$h(1) = h_d(1)W_H(1) = 0.0469 \times 0.2147 = 0.0100$$

$$h(2) = h_d(2)\omega_H(2) = -0.1075 \times 0.54 = -0.0580$$

$$h(3) = h_d(3)\omega_H(3) = -0.2966 \times 0.8652 = -0.2566$$

$$h(4) = h_d(4)\omega_H(4) = 0.618 \times 1 = 0.618$$

$$h(5) = h_d(5)\omega_H(5) = -0.2966 \times 0.8652 = -0.2566$$

$$h(6) = h_d(6)\omega_H(6) = -0.1075 \times 0.54 = -0.0580$$

$$h(7) = h_d(7)\omega_H(7) = 0.0469 \times 0.2147 = -0.0100$$

$$h(8) = h_d(8)\omega_H(8) = 0.0792 \times 0.08 = -0.0063$$

From the above calculations, we can observe that $h(N-1-n) = h(n)$ i.e, the impulse response is symmetrical with centre of symmetry at $n=4$.

The frequency response of the filter is

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{8} h(n) e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega} + h(5)e^{-j5\omega} + h(6)e^{-j6\omega}$$

$$+ h(7)e^{-j7\omega} + h(8)e^{-j8\omega}$$

$$= e^{-j4\omega}\left[h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega \right]$$

$$= e^{-j4\omega}\left[0.618 + 2(-0.256)\cos\omega + 2(-0.058)\cos 2\omega + 2(0.010)\cos 3\omega \right.$$

$$\left. + 2(0.0063)\cos 4\omega \right]$$

$$= e^{-j4\omega}\left[0.618 - 0.5132\cos\omega - 0.116\cos 2\omega + 0.020\cos 3\omega + 0.012\cos 4\omega \right]$$

The magnitude response is given by

$$|H(\omega)| = 0.1618 - 0.5132\cos\omega - 0.116\cos 2\omega + 0.020\cos 3\omega + 0.012\cos 4\omega$$

The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{8} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

$$= 0.0063 + 0.010z^{-1} - 0.0580z^{-2} - 0.2566z^{-3} + 0.618z^{-4} - 0.2566z^{-5}$$

$$-0.0580z^{-6} + 0.0100z^{-7} + 0.0063z^{-8}$$

$$= z^{-4}\left[0.618 - 0.2566(z+z^{-1}) - 0.0580(z^{2}+z^{-2}) + 0.0100(z^{3}+z^{-3}) \right.$$

$$\left. + 0.0063(z^{4}+z^{-4}) \right]$$

Normalized Ideal frequency response and Impulse response for FIR filter design using Windows.

Type of the filter	Ideal (desired) frequency response	Ideal (desired) Impulse response.
1. Low-Pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & ; -\omega_c \le \omega \le \omega_c \\ 0 & ; -\pi \le \omega \le -\omega_c \\ 0 & ; \omega_c \le \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\,d\omega$ $= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} \cdot e^{j\omega n}\,d\omega$ $= \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$
2. High-Pass filter	$H_d(\omega) = e^{-j\alpha\omega}$; $-\pi \le \omega \le -\omega_c$ $e^{-j\alpha\omega}$; $\omega_c \le \omega \le \pi$ 0 ; $-\omega_c \le \omega \le \omega_c$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\,d\omega$ $= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} \cdot e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} \cdot e^{j\omega n}\,d\omega$ $= \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$
3. Band-Pass filter	$H_d(\omega) = e^{-j\omega\alpha}$; $-\omega_{c_2} \le \omega \le -\omega_{c_1}$ $= e^{-j\omega\alpha}$; $\omega_{c_1} \le \omega \le \omega_{c_2}$ $= 0$; $-\pi \le \omega \le -\omega_{c_2}$ $= 0$; $-\omega_{c_1} \le \omega \le \omega_{c_1}$ $= 0$; $\omega_{c_2} \le \omega \le \pi$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{-j\omega n}\,d\omega$ $= \frac{1}{2\pi}\int_{-\omega_{c_2}}^{\omega_{c_2}} e^{-j\omega\alpha} \cdot e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{+\omega_{c_1}}^{\omega_{c_2}} e^{-j\omega\alpha} \cdot e^{j\omega n}\,d\omega$ $= \frac{\sin\omega_{c_2}(n-\alpha) - \sin\omega_{c_1}(n-\alpha)}{\pi(n-\alpha)}$
4. Band-stop filter	$H_d(\omega) = e^{-j\omega\alpha}$; $-\pi \le \omega \le -\omega_{c_2}$ $e^{-j\omega\alpha}$; $-\omega_{c_1} \le \omega \le \omega_{c_1}$ $e^{-j\omega\alpha}$; $\omega_{c_2} \le \omega \le \pi$ 0 ; $-\omega_{c_2} \le \omega \le -\omega_{c_1}$ 0 ; $\omega_{c_1} \le \omega \le \omega_{c_2}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\,d\omega$ $= \frac{\sin\omega_{c_1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c_2}(n-\alpha)}{\pi(n-\alpha)}$

Design of FIR FILTERS BY FREQUENCY SAMPLING TECHNIQUE

In this method, the ideal frequency response is sampled at sufficient no. of points (i.e N-points). These samples are the DFT co-efficients of the impulse response of the filter. Hence the impulse response of the filter is determined by taking IDFT.

Let $H_d(\omega)$ = Ideal (desired) frequency response

$\tilde{H}(k)$ = The DFT sequence obtained by sampling $H_d(\omega)$

$h(n)$ = Impulse response of FIR filter.

Procedure

1. choose the ideal (desired) frequency response $H_d(\omega)$

2. sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = \frac{2\pi k}{N}$, where $k = 0, 1, 2, \ldots (N-1)$ to generate the sequence $\tilde{H}(k)$

$$\therefore \tilde{H}(k) = H_d(\omega)\Big|_{\omega = (2\pi k/N)} \quad \text{for } k = 0, 1, 2 \ldots N-1$$

3. Compute the N-samples of $h(n)$ using the following equations:

When N is odd,
$$h(n) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{\left(\frac{N-1}{2}\right)} Re\left(\tilde{H}(k)e^{j2\pi k/N}\right)\right]$$

When N is even,
$$h(N) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{\left(\frac{N}{2}-1\right)} \left(\tilde{H}(k)e^{j2\pi nk/N}\right)\right]$$

where 'Re' stands for 'real part of'

4. Take z-transform of the impulse response $h(n)$ to get the transfer function $H(z)$

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

→ Comparision of FIR and IIR filters

IIR Filters	FIR Filters
1. All the infinite samples of impulse response are considered	1. Only a finite no. of samples of impulse response are considered.
2. The impulse response cannot be directly converted to digital filter transfer function.	2. The impulse response can be directly converted to digital filter transfer function.
3. Linear phase characteristics cannot be achieved	3. Linear phase filters can be easily designed.
4. IIR filters are easily realized recursively	4. FIR filters can be realized recursively and non-recursively
5. The specifications include the desired characteristics for magnitude response only	5. The specifications include the desired characteristics for both magnitude and phase response
6. The design involves design of analog filter and then transforming analog filter to digital filter	6. The digital filter can be directly designed to achieve the desired specifications.
7. The round off noise in IIR filters is more.	7. Errors due to round off noise are less severe in FIR filters, mainly feedback is not used
8. Less flexibility, usually limited to specific kind of filters	8. Greater flexibility to control the shape of their magnitude response.

Realization of FIR Filters

Transversal structure (or) Direct form realization

The system function of an FIR filter can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(N-1)z^{-(N-1)}$$

$$Y(z) = h(0)X(z) + h(1)z^{-1}X(z) + h(2)z^{-2}X(z) + \dots + h(N-1)z^{-(N-1)}X(z) \quad \text{—(1)}$$

The above equation can be realized as shown in fig

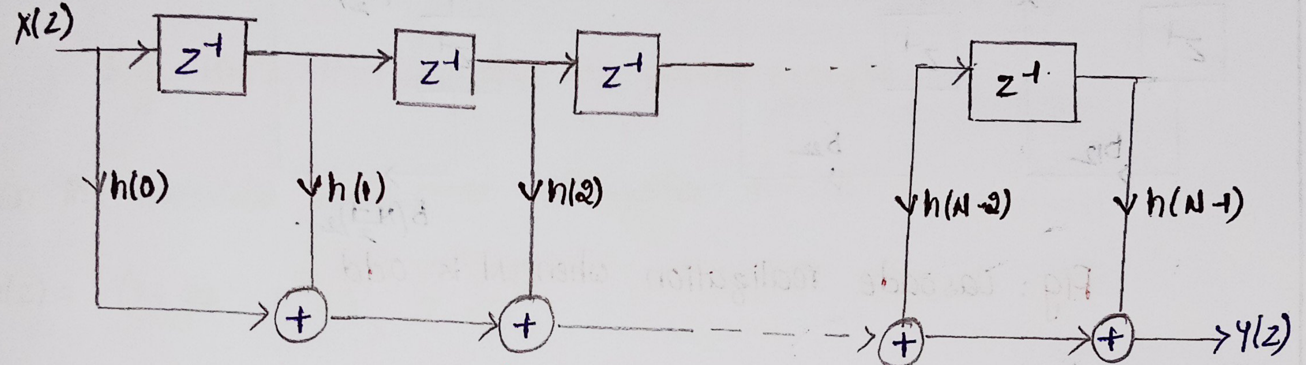


Fig: Direct form realization of eq (1)

This structure is known as transversal structure or direct form realization. The transversal structure requires N multipliers, N-1 adders and N-1 delay elements.

Cascade Realization

The system function of an FIR filter can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(N-1)z^{-(N-1)}$$

The above equation can be realized in cascade form the factored form of H(z). For N odd

$$H(z) = \prod_{k=1}^{\left(\frac{N-1}{2}\right)} \left(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}\right)$$

$$= (b_{10} + b_{11}z^{-1} + b_{12}z^{-2})(b_{20} + b_{21}z^{-1} + b_{22}z^{-2}) \ldots \ldots (b_{(\frac{N-1}{2})0} + b_{(\frac{N-1}{2})1}z^{-1} + b_{(\frac{N-1}{2})2}z^{-2})$$

For N odd, N-1 will be even and H(z) will have $(\frac{N-1}{2})$ second order factors. Each second order factored form of H(z) is realized in direct form and is cascaded to realize H(z) as shown in below figure.

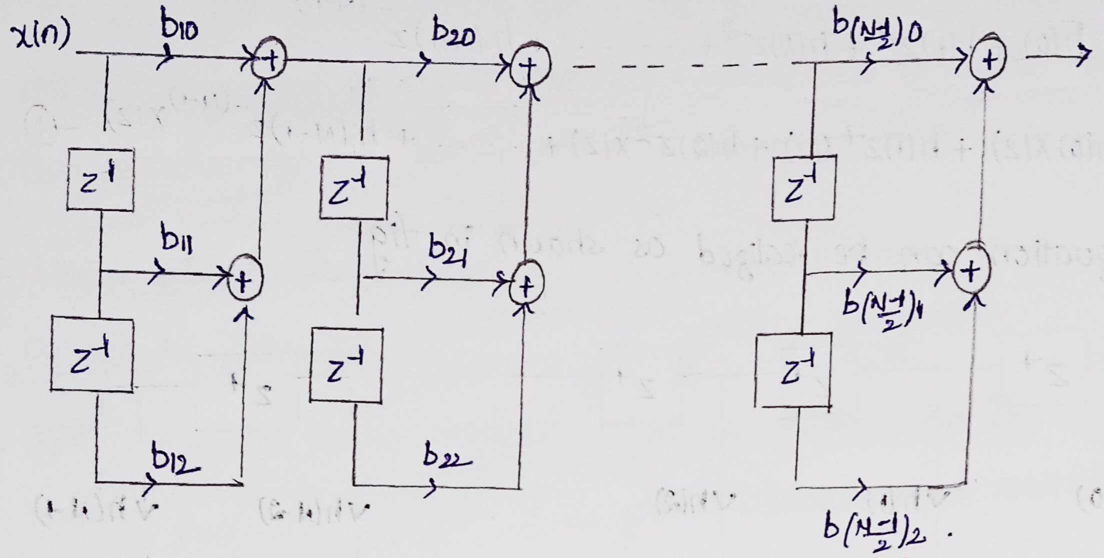


Fig: Cascade realization when N is odd

For N even

$$H(z) = (b_{10} + b_{11}z^{-1}) \prod_{k=2}^{N/2} (b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})$$

when N is even, N-1 is odd and H(z) will have one first order factor and $(\frac{N-2}{2})$ second order factors

$$H(z) = (b_{10} + b_{11}z^{-1})(b_{20} + b_{21}z^{-1} + b_{22}z^{-2})(b_{30} + b_{31} + b_{32}z^{-2}) \ldots (b_{\frac{N}{2}0} + b_{(\frac{N}{2})1}z^{-1} + b_{(\frac{N}{2})2}z^{-2})$$

Now each factored form in H(z) is realized in direct form and is cascaded to obtain the realization of H(z) as shown in below figure

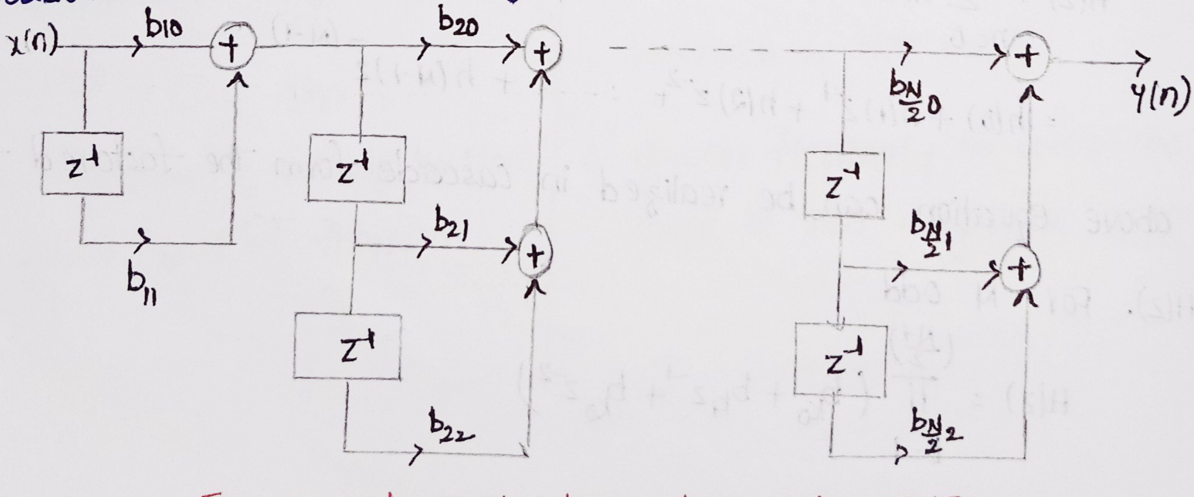


Fig: Cascade realization when N is EVEN

→ Determine the direct form realization of system function

$H(z) = 1+2z^{-1}-3z^{-2}-4z^{-3}+5z^{-4}$

Sol. Given $H(z) = 1+2z^{-1}-3z^{-2}-4z^{-3}+5z^{-4}$

The above equation can be realized in direct form-I of FIR filter as shown in fig:

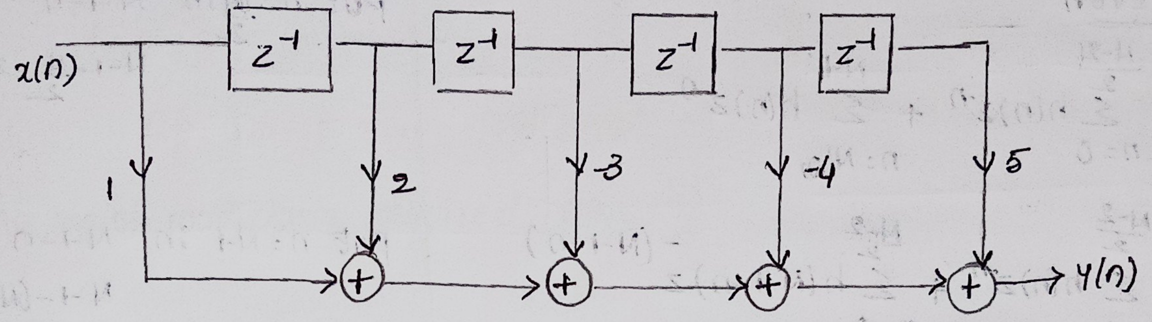


Fig: Realization structure of given example.

→ Obtain the cascade realization of system function

$H(z) = (1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$

Sol Given $H(z) = (1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$

where $H_1(z) = (1+2z^{-1}-z^{-2})$ $\quad H_2(z) = (1+z^{-1}-z^{-2})$

$H_1(z) = \frac{Y_1(z)}{X_1(z)} = 1+2z^{-1}-z^{-2}$

$Y_1(z) = X_1(z)+2z^{-1}X_1(z)-z^{-2}X_1(z).$

$H_2(z) = 1+z^{-1}-z^{-2} = \frac{Y_2(z)}{X_2(z)}.$

$Y_2(z) = X_2(z)+z^{-1}X_2(z)-z^{-2}X_2(z).$

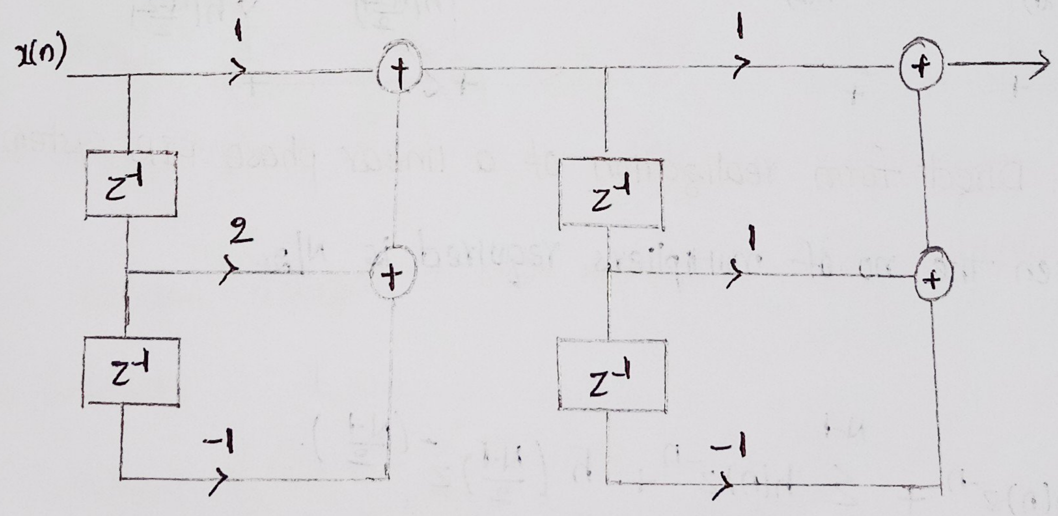


Fig: Cascade realization of given example.

→ Linear Phase Realization

For a linear Phase FIR filter

$$h(n) = h(N-1-n)$$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Limits

$h(n) = h(N-1-n)$

put $n = \frac{N}{2}$ in $N-1-n$

$N-1-\frac{N}{2} = \frac{2N-2-N}{2}$

$= \frac{N-2}{2}$

put $n = N-1$ in $N-1-n$

$N-1-(N-1) = 0$

For N is even

$$H(z) = \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-n} + \sum_{n=N/2}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n) z^{-(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-n} + \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) \left[z^{-n} + z^{-(N-1-n)} \right]$$

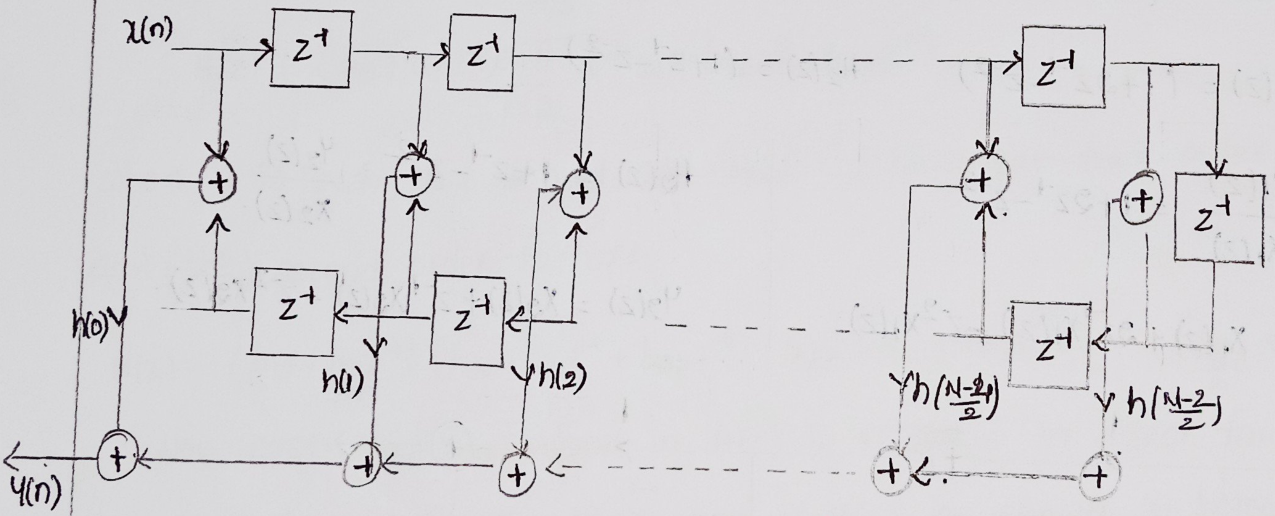


Fig: Direct form realization of a linear phase FIR system for N even

For N is even the no. of multipliers required is $N/2$.

For N is odd

$$H(z) = \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)}.$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[z^{-n} + z^{-(N-1-n)} \right]$$

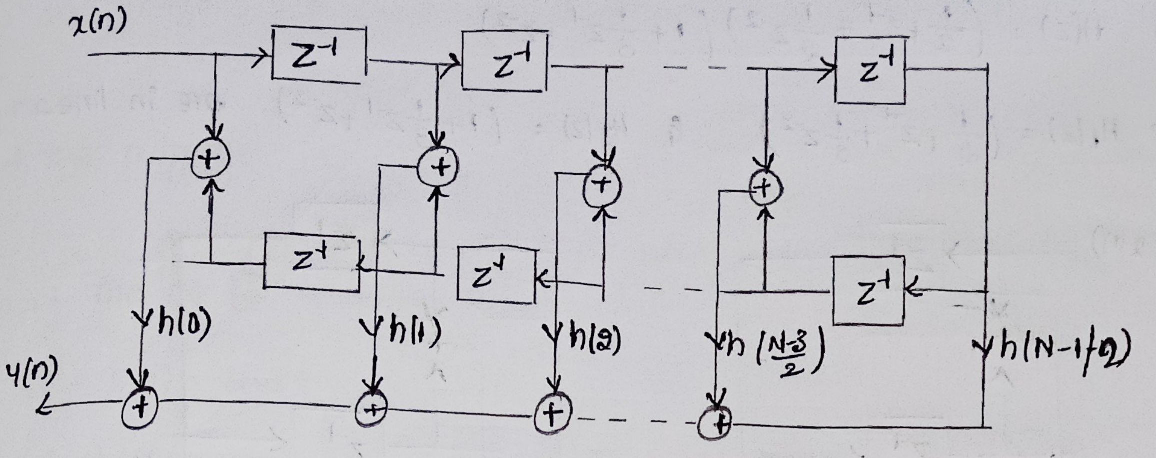


Fig: Direct form realization of a linear phase FIR system for N odd

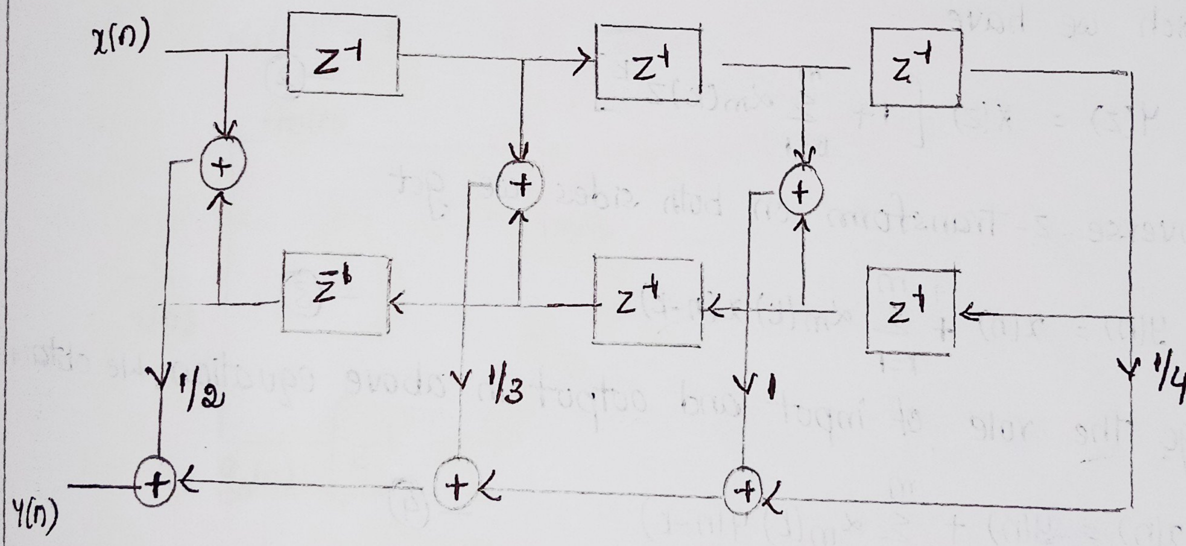
The no. of multipliers required are $\frac{N+1}{2}$

→ Realize the system function

$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$$

Sol: By inspection we find that the system function H(z) is that of a linear Phase FIR filter and $h(N) = h(N-1-n)$

Given difference equation i.e system function length is odd $(N=7)$.



→ obtain cascade realization with minimum no. of multiplies for the system function

$$H(z) = \left(\frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2}\right)\left(1 + \frac{1}{3}z^{-1} + z^{-2}\right)$$

Sol. Given $H(z) = \left(\frac{1}{2}+z^{-1}+\frac{1}{2}z^{-2}\right)\left(1+\frac{1}{3}z^{-1}+z^{-2}\right)$

where $H_1(z) = \left(\frac{1}{2}+z^{-1}+\frac{1}{2}z^{-2}\right)$ & $H_2(z) = \left(1+\frac{1}{3}z^{-1}+z^{-2}\right)$ are in linear phase

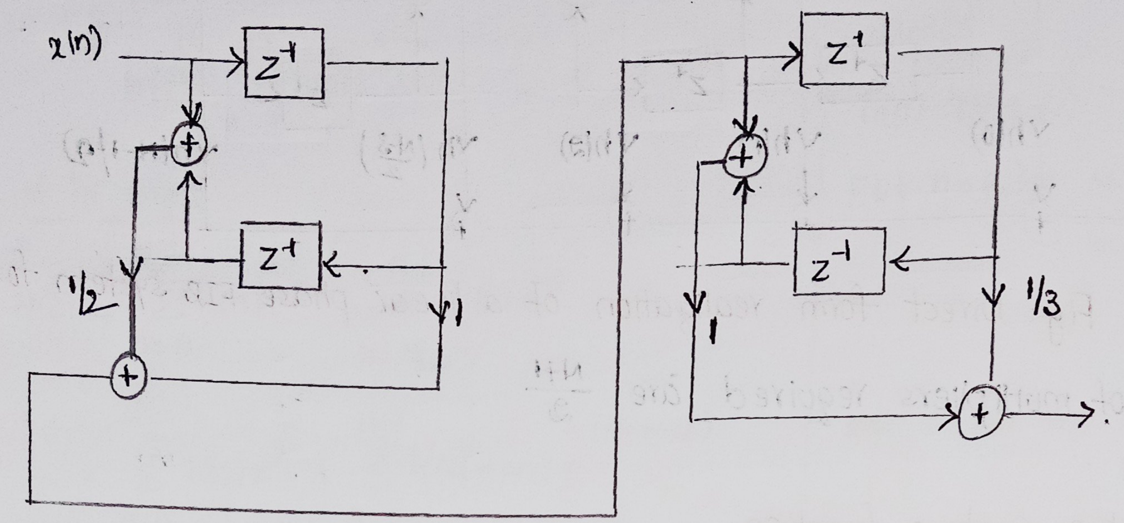


Fig: Cascade realization of given example.

→ Lattice structure of an FIR filter

Let us consider an FIR filter with system function

$$H(z) = A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k} \qquad m \geq 1 \qquad -①$$

from which we have

$$Y(z) = X(z)\left[1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k}\right] \qquad -②$$

Taking inverse z-transform on both sides we get

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)\, x(n-k) \qquad -③$$

Interchange the role of input and output in above equation, we obtain

$$x(n) = y(n) + \sum_{k=1}^{m} \alpha_m(k)\, y(n-k) \qquad -④$$

We find that the eq ④ describes an IIR system having the system function $H(z) = \frac{1}{A_m(z)}$, while the system described by the difference equation in eq ③ represents an FIR system with system function $H(z) = A_m(z)$.

For an all-zero FIR system of order M-1 the input

$$x(n) = f_0(n)$$

and the output

$$y(n) = f_{M-1}(n)$$

For $m = 1$ the eq ③ reduces to

$$y(n) = x(n) + \alpha_1(1)\, x(n-1) \quad — ⑤$$

This output can also be obtained from a single-stage lattice filter shown in fig 1 from which we have

$$x(n) = f_0(n) = g_0(n)$$

$$y(n) = f_1(n) = f_0(n) + k_1 g_0(n-1)$$

$$y(n) = x(n) + k_1 x(n-1) \quad — ⑥$$

and

$$g_1(n) = k_1 f_0(n) + g_0(n-1)$$

$$= k_1 x(n) + x(n-1) \quad — ⑦$$

Comparing eq ④ with eq ⑦, we get

$$\alpha_1(0) = 1, \quad \alpha_1(1) = k_1$$

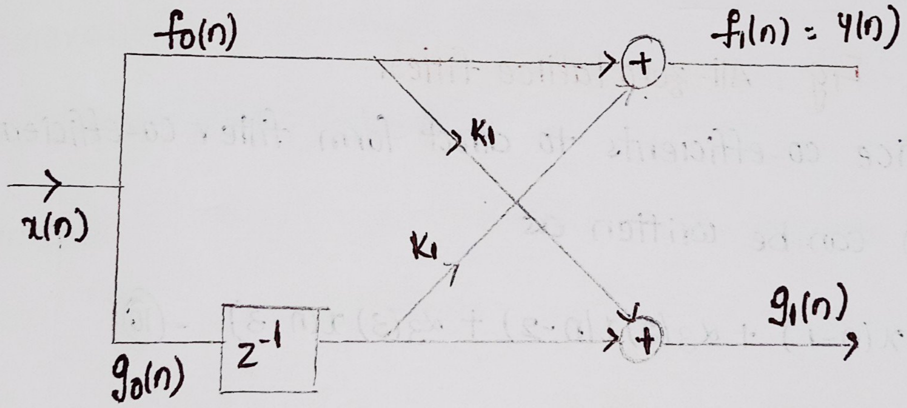
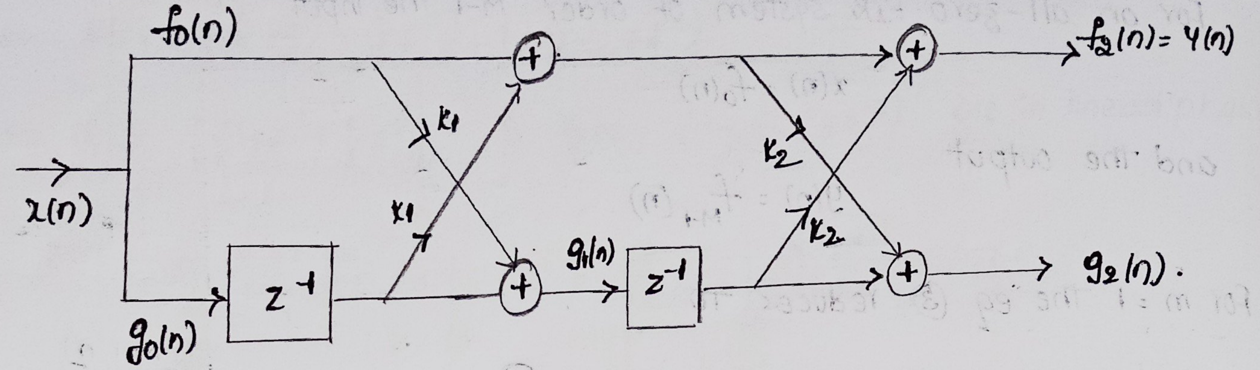


Fig: single stage all-zero lattice filter

Now, let us consider an FIR filter for which $m=2$, then

$$y(n) = x(n) + \alpha_2(1)\, x(n-1) + \alpha_2(2)\, x(n-2)$$

By cascading two lattice stages as shown in fig 2, it is possible to obtain the output $y(n)$



For a M-1 stage filter

$$f_0(n) = g_0(n)$$

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1) \qquad m = 1,2 \ldots M-1 \quad -(8)$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \qquad m = 1,2, \ldots M-1 \quad -(9)$$

The output of M-1 stage filter $y(n) = f_{M-1}(n)$.

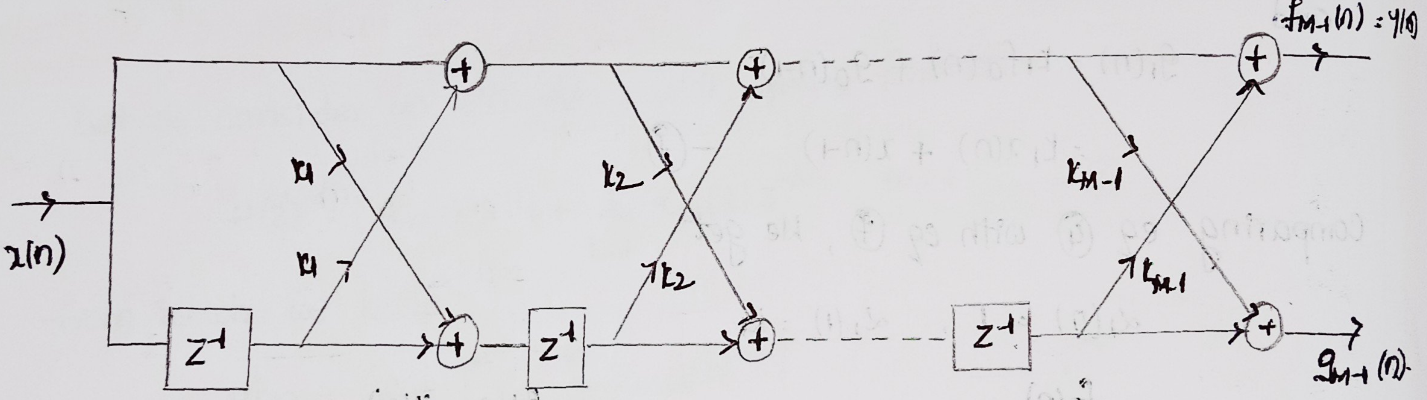


Fig : All-zero lattice filter.

Conversion of lattice co-efficients to direct form filter co-efficients

For $m=3$ the eq (3) can be written as

$$y(n) = x(n) + \alpha_3(1) x(n-1) + \alpha_3(2) x(n-2) + \alpha_3(3) x(n-3) \quad -(10)$$

We have

$$y(n) = f_3(n) = f_2(n) + K_3 g_2(n-1)$$

$$= x(n) + \alpha_2(1) x(n-1) + \alpha_2(2) x(n-2) + K_3 \alpha_2(2) x(n-1)$$

$$+ K_3 \alpha_2(1) x(n-2) + K_3 x(n-3)$$

$$= x(n) + [\alpha_2(1) + K_3 \alpha_2(1)] x(n-1) + [\alpha_2(2) + K_3 \alpha_2(1)] x(n-2)$$

$$+ K_3 x(n-3) \quad -(11)$$

Comparing the eq ⑩ & ⑪, we get

$$\alpha_3(0) = 1$$

$$\alpha_3(1) = \alpha_2(1) + k_3\alpha_2(2)$$

$$= \alpha_2(1) + \alpha_3(3)\alpha_2(2)$$

$$\alpha_3(2) = \alpha_2(2) + k_3\alpha_2(1)$$

$$= \alpha_2(2) + \alpha_3(3)\alpha_2(1)$$

$$\alpha_3(3) = k_3$$

For a General case, we find that

$$\alpha_m(0) = 1$$

$$\alpha_m(m) = k_m$$

$$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-k)$$

The above equation can be used to convert the lattice co-efficients to direct form FIR filter co-efficients.

Conversion of direct form FIR filter co-efficients to lattice co-efficients.

In general for a m-stage filter

$$\alpha_{m-1}(0) = 1, \quad k_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)} \qquad 1 \le k \le m-1$$

→ An FIR filter is given by the difference equation

$$y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Determine its lattice form.

Sol: Given

$$y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

$$= 2\left[x(n) + \frac{2}{5}x(n-1) + \frac{3}{4}x(n-2) + \frac{1}{3}x(n-3)\right]$$

$$= K_0\left[1 + \sum_{k=1}^{3} \alpha_m(k)\, x(n-k)\right]$$

where $K_0 = 2$, $\alpha_3(1) = \frac{2}{5}$, $\alpha_3(2) = \frac{3}{4}$, $\alpha_3(3) = \frac{1}{3}$, $\alpha_3(0) = 1$.

We know that

$$\alpha_{m-1}(0) = 1 \Rightarrow \alpha_2(0) = 1$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\,\alpha_m(m-k)}{1 - \alpha_m^2(m)}$$

For $m=3$, $k=1$

$$\alpha_2(1) = \frac{\alpha_3(1) - \alpha_3(3)\,\alpha_3(2)}{1 - \alpha_3^2(3)} = \frac{\frac{2}{5} - \frac{1}{3}\cdot\frac{3}{4}}{1 - \frac{1}{9}} = \frac{\frac{2}{5} - \frac{1}{4}}{\frac{8}{9}} = 0.16875$$

For $m = 3$ & $k = 2$

$$K_2 = \alpha_2(2) = \frac{\alpha_3(2) - \alpha_3(3)\,\alpha_3(1)}{1 - \alpha_3^2(3)}$$

$$= \frac{\frac{3}{4} - \frac{1}{3}\cdot\frac{2}{5}}{1 - \left(\frac{1}{3}\right)^2} = \frac{\frac{3}{4} - \frac{2}{15}}{1 - \frac{1}{9}} = 0.69375$$

For $m = 2$ & $k = 1$

$$K_1 = \alpha_1(1) = \frac{\alpha_2(1) - \alpha_2(2)\,\alpha_2(1)}{1 - \alpha_2^2(2)} = \frac{\frac{27}{160} - \frac{111}{160}\cdot\frac{27}{160}}{1 - \left(\frac{111}{160}\right)^2}$$

$$= 0.0996$$

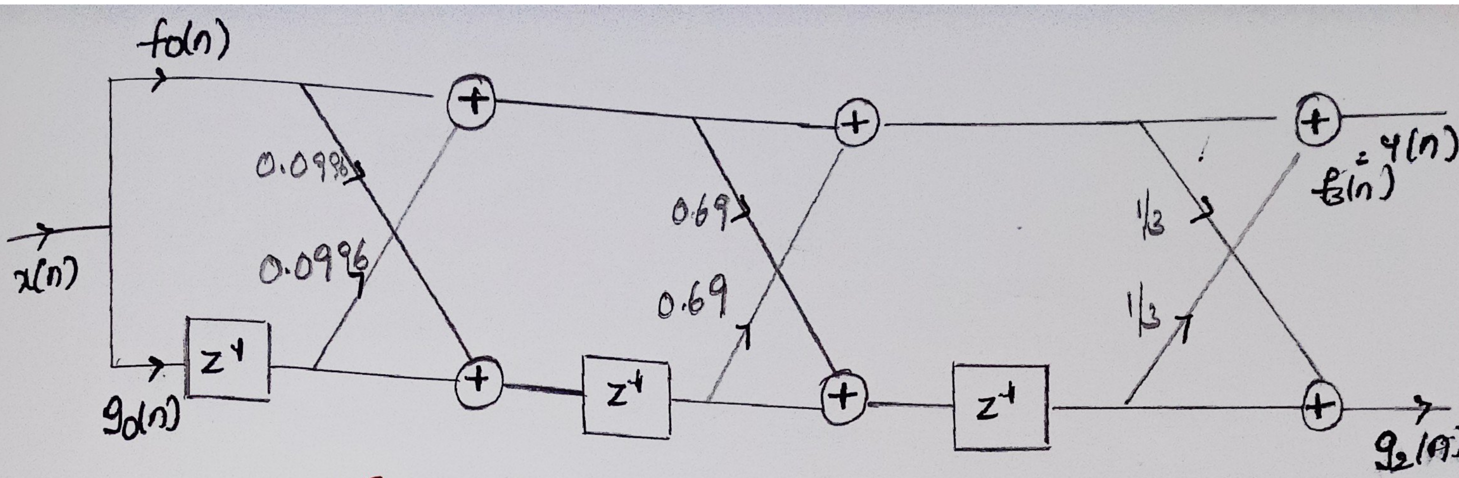


Fig: Lattice structure of given example